Three-dimensional *x*-*y* model with a Chern-Simons term

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(Received 26 September 1995)

We introduce a model where a Chern-Simons term is coupled to the three-dimensional *x*-*y* model. This term endows vortices with an internal angular momentum and thus they acquire arbitrary statistical character. In our interpretation of the Chern-Simons term in the three-dimensional *x*-*y* model, it takes an integer value that can be written as a sum over all vortex lines of the product of the vortex charge and the winding number of the internal phase angle along that vortex line. We have used the Monte Carlo method to study the three-dimensional *x*-*y* model with the Chern-Simons term. Our findings suggest that this model belongs to the *x*-*y* universality class with the critical temperature growing with increasing internal angular momentum.

I. INTRODUCTION

A phenomenological theory, similar to the Landau-Ginzburg theory, which describes aspects of the integer and fractional quantum Hall effect is the Chern-Simons-Landau-Ginzburg (CSLG) theory.[1–4] In such a description, the electrons are minimally coupled to a statistics changing "electromagnetic" field whose action is the topological Chern-Simons term. The CSLG theory has been used by several authors to study the transitions from a quantum Hall liquid to the quantum Hall insulator[5,6] and the transitions between different quantum Hall liquids, i.e., between different quantum Hall plateaus. Jain, Kivelson, and Trivedi advanced the idea that the transitions between different quantum Hall states belong to the same universality class (within a certain family),[7] i.e., the critical exponents characterizing these transitions are independent of the filling factor and thus of the statistics of the quasiparticles. This argument can also be made using duality considerations.[1]

On the other hand, there have been studies of relativistic versions (where disorder is completely absent) of the Chern-Simons-Landau-Ginzburg theory which attempt to determine the dependence of the critical properties on the coefficient in front of the Chern-Simons term.[8–12] The authors of Refs. 9 and 10 study the O(N) model and use a $1/N$ expansion, where N is the number of components of the order parameter, to compute the critical exponents. They find that the critical exponents depend on the coefficient in front of the Chern-Simons term. The authors of Ref. 11 consider the Dirac theory in three dimensions and perform a perturbative expansion in the Chern-Simons coefficient to find the critical exponents of the theory to vary with this coefficient. In Ref. 12 an expansion up to one loop of the scalar ϕ^4 theory coupled to the Chern-Simons term is used to study the critical properties of this theory. It was found that the fluctuations in the gauge field reduce the order of the transition from second order in the absence of the gauge field to first order in the presence of these fluctuations. We remark that the scalar ϕ^4 theory without the gauge field and close to the critical point is equivalent to the O(2) or *x*-*y* model. Therefore the question as to whether and to what extent the statistics of the quasiparticles change the critical properties of a physical system is still controversial.

This problem can be studied further by numerical investigation of the critical properties of a relatively simple model coupled to the Chern-Simons term. In this paper we couple the three-dimensional *x*-*y* model [or O(2) model] to the Chern-Simons term following the approach of Wilczek and Zee[13] who studied the nonlinear sigma [O(3) invariant model] coupled to a Hopf term. We investigate how this topological Chern-Simons term influences the critical properties of this new model and we study whether or not they are different from the critical properties of the *x*-*y* model itself.

We wish to emphasize here that there are two ways of introducing the Chern-Simons term in the theory; first, it can be introduced by minimally coupling a fluctuating U(1) gauge field to the complex matter field ϕ where the action of the gauge field is just the Chern-Simons term.[12,14] In this case the field quanta acquire different statistics in the long-wavelength limit. Second, one can define a U(1) gauge field through a conserved current and add the Chern-Simons term for this gauge field to the Lagrangian.[13,15,16] Then extended particles such as skyrmions in the O(3) model[13] and vortices in the *x*-*y* model acquire different statistics.

The *x*-*y* model without the Chern-Simons term in two and three dimensions has been studied extensively (cf., e.g., Ref. 17). It was found that vortex excitations play a crucial role in the phase transition. Kosterlitz and Thouless showed that the pairing of vortices in the *x*-*y* model causes a phase transition in two dimensions.[18] In three dimensions the vortex strings seem responsible for the occurrence of the phase transition,[19] which has the character of an order-disorder phase transition. The remarkable property of the *x*-*y* model is that the topological objects, vortices in two and vortex lines in three dimensions, which are responsible for the occurrence of the phase transition, do not explicitly appear in the model. They are rather created dynamically and extended over long length scales. Thus we can consider the *x*-*y* model as a microscopic field theory that contains quasiparticles that are extended objects over long length scales and interact through a potential

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created by the interaction of the microscopic degrees of freedom. In the two-dimensional *x*-*y* model the quasiparticles are the vortices that interact through a logarithmic potential, while the microscopic degrees of freedom are elementary and are coupled by a nearest-neighbor potential (cf. the next section). The three-dimensional *x*-*y* model can then be viewed as describing two-dimensional vortices moving in Euclidean time. By coupling the three-dimensional *x*-*y* model to the Chern-Simons term we can give the vortices an internal angular momentum taking values between 0 and 1/2. Since the internal angular momentum of the particles is closely related to statistics, quasiparticles whose statistical character can be interpolated between that of a boson and a fermion, are dynamically created.

In order to introduce the Chern-Simons term in the three-dimensional *x*-*y* model we need to identify a current. The natural choice in this case is the vortex current that, since it is represented by the vortex lines, can be easily identified. Since the vortex current is conserved we can write it as the curl of a vector potential $\vec{A}$, which enters into the definition of the Chern-Simons term. It turns out that each single vortex line gives a contribution to the Chern-Simons term which is the product of the vortex charge and the winding number of the internal phase angle (cf. the next section) along that vortex line. This interpretation of the Chern-Simons term enables us to compute this term for an arbitrary lattice field configuration. We then couple the Chern-Simons term to the *x*-*y* model and compute the helicity modulus and the specific heat for this new model. The finite-size analysis of these quantities leads us to the conclusion that the *x*-*y* model with the Chern-Simons term exhibits the same critical properties as the *x*-*y* model itself.

The details concerning these introductory remarks are described in the following sections. In the next section we introduce the three-dimensional *x*-*y* model and derive the expression for the Chern-Simons term for this model. The partition function for the *x*-*y* model with the Chern-Simons term is also given. In Sec. III we describe the physical observables such as the helicity modulus and the specific heat and discuss the numerical method, which we employed to compute these quantities. Our results are given in Sec. IV. In the last section we briefly summarize our findings.

II. THE CHERN-SIMONS TERM IN THE *x*-*y* MODEL

In this section we define the *x*-*y* model and give a geometrical interpretation of the Chern-Simons term for this model.

The partition function of the *x*-*y* model on the lattice is defined as

$$Z=\int \prod_j d\theta_j \exp(-\beta\mathscr{H}), \tag{1}$$

where $\beta=J/k_BT$ and

$$\mathscr{H}=-\sum_{\langle i,j\rangle} \vec{s}_i\cdot\vec{s}_j . \tag{2}$$

The sum in (2) runs over nearest neighbors, $\vec{s}_i=(\cos\theta_i,\sin\theta_i)$ is a two-component vector, which is constrained to be on the unit circle and J sets the energy scale.

Before we turn to the construction of the Chern-Simons term for the *x*-*y* model let us briefly discuss the case of the O(3) model. This model is different from the *x*-*y* model only in that the spin space is a unit sphere in three dimensions. The equation of motion of the O(3) model admits soliton solutions also known as skyrmions. Wilczek and Zee[13] showed that the skyrmions can be given arbitrary statistics if the Langrangian of the O(3) model is coupled to the Chern-Simons term. Some of the arguments of Wilczek and Zee can be applied to the case of the *x*-*y* model in a straightforward manner. Furthermore we will make use of the "form" language (see, for example, Ref. 20) because this formalism is very convenient for our purposes.

The Chern-Simons term H_{CS} for an Abelian gauge field A is defined as follows:

$$H_{\mathrm{CS}}=k\int A\wedge dA, \tag{3}$$

where

$$A=A_\mu dx^\mu, \tag{4}$$

$$dA=\partial_\nu A_\mu dx^\nu\wedge dx^\mu. \tag{5}$$

The constant k in (3) will be specified later. The symbols in Eqs. (3)–(5) and in the following are defined as in Ref. 20.

Let us consider a spin configuration with q vortex lines on a L^3 lattice with periodic boundary conditions. Each vortex line represents a vortex of an integer vortex charge $n_q\in I$. Due to the periodic boundary conditions all vortex lines are closed. If we travel along a vortex line the spins forming the vortex may rotate in the internal spin space. This vortex line is topologically different from a vortex line, representing the same vortex charge but with a different number of spin rotations. We expect the Chern-Simons number (3) to contain this information.

Given a conserved current j a gauge field A can be constructed by[13]

$$dA=\star j. \tag{6}$$

In our case j is the current 1-form whose support is the vortex lines. We write

$$j=\sum_q j_q, \tag{7}$$

where j_q denotes the qth individual vortex current. The vortex charge n_q is obtained from

$$\int_{\partial C_q} d\theta=2\pi n_q, \tag{8}$$

∂C_q is an arbitrary contour enclosing the core of the qth vortex line, θ is the spin angle. Note also that

$$\int_{C_q}\star j=2\pi n_q, \tag{9}$$

where C_q is the surface bounded by the curve ∂C_q. We want the gauge field A to satisfy the following condition:

$$\int_{\partial C_q} A = 2\pi n_q , \tag{10}$$

or equivalently by the Gauss-Bonnet theorem

$$\int_{C_q} dA = 2\pi n_q . \tag{11}$$

Let us try the ansatz

$$A = d\theta + \sum_q \omega_q , \tag{12}$$

where the support of the 1-forms ω_q is also restricted to the vortex lines. Using

$$d\omega_q = \star j_q \tag{13}$$

we are able to satisfy (10) and (11).

Equation (3) takes the following form now:

$$H_{\mathrm{CS}} = k \int \left(d\theta + \sum_q \omega_q \right) \wedge \sum_{q'} d\omega_{q'} \tag{14}$$

$$= k \sum_q \int d\theta \wedge d\omega_q + k \sum_q \int \omega_q \wedge d\omega_q . \tag{15}$$

Let us spread out the vortex lines, i.e., we define the support of the forms ω_q to be a thin tube of radius $\epsilon \rightarrow 0$ around the vortex lines. Introducing a comoving 3-bein as a local coordinate system to express ω_q locally, we realize that ω_q has components only in a disk perpendicular to its vortex line, or in other words, the last term in (13) vanishes. We are left with

$$H_{\mathrm{CS}} = k \sum_q \int d\theta \wedge d\omega_q \tag{16}$$

or, according to the Poincare duality, with

$$H_{\mathrm{CS}} = k \sum_q 2\pi n_q \int_{l_q} d\theta, \tag{17}$$

where we integrate along the qth vortex line now. Since θ is not defined at the center of a vortex we have to give the integral

$$\int_{l_q} d\theta \tag{18}$$

a well defined meaning. Namely, instead of integrating along the actual vortex line we integrate along a line infinitesimally shifted from the original vortex line. The integral (18) measures the spin rotations m_q, and we obtain

$$H_{\mathrm{CS}} = k 4\pi^2 \sum_q n_q m_q . \tag{19}$$

If we choose the constant k in (3) to be $1/4\pi^2$, we can define the Chern-Simons number N_{CS} as

$$N_{\mathrm{CS}} = \sum_q n_q m_q . \tag{20}$$

From here we can immediately see that N_{CS} changes its sign under parity or time-inversal transformations. Let us consider the z axis as the Euclidean time axis. Then we can think of our model as describing vortices moving in time. A closed loop corresponds to the creation of a vortex and antivortex that move apart and are annihilated after some time. Time-inversion inverses the path in the integral (16), i.e., changes the sign of m_q, but leaves the vortex charge n_q unchanged, whereas a parity transformation inverses the path in the integral (10), i.e., changes the sign of n_q, but leaves the spin rotations m_q unchanged.

In order to determine the Chern-Simons number N_{CS} for a configuration of spins $\{\vec{s}_i\}$ on the lattice we need to measure the spin rotations m_q along the vortex lines. The vortex lines are perpendicular to the plaquettes and can be found by making use of the conservation of the vortex current, if a cube has an ingoing vortex current, it has an outgoing vortex current as well. The spin rotations m_q can be measured as follows. After introducing a comoving 3-bein that moves along the vortex line and fixing the position of an arbitrary spin with respect to the 3-bein, we determine how often a spin at our fixed position rotates when we travel along the vortex line.

In the presence of the Chern-Simons term H_{CS} we define the partition function of the x-y model as follows:[13]

$$Z = \int \prod_j d\theta_j \exp(-\beta \mathscr{H} + i\alpha N_{\mathrm{CS}}). \tag{21}$$

Since we expect the probabilities to find spin configurations with $+N_{\mathrm{CS}}$ and $-N_{\mathrm{CS}}$ to be equal we restrict the angle variable α to the interval $\alpha \in [0,\pi]$. Let us now consider a vortex with charge ± 1. If we rotate this vortex adiabatically through 2π over a long period of time the contribution to the partition function (21) is $\exp(i\alpha)$, i.e., the vortex has an internal angular momentum $\alpha/2\pi$. For $\alpha = \pi$ this vortex behaves like a fermion. Since we identify a vortex on a plaquette by computing the winding number of the phase angle θ around that plaquette according to Eq. (8) the vortices have only charges ± 1; we can give these vortices an arbitrary internal angular momentum or spin depending on the value of α. If we again think of our model as describing vortices moving in Euclidean time and if we only consider vortices of charge ± 1, it seems tempting to assume that now the lowest-energy state consists of pairs of vortices with opposite charge and opposite spin. This pairing mechanism has its origin in the interaction between the vortices and the additional interaction between the spins. Thus an attractive potential due to the opposite spins in addition to the attractive potential due to the opposite vortex charges between the two paired vortices is introduced. As an intuitive example we can consider two particles of opposite charge moving in a plane. If the particles have spins $\vec{\mu}_1$ and $\vec{\mu}_2$, respectively, in addition to the logarithmic attraction due to the opposite charges the particles feel the interaction energy $V(r) = f(r) \vec{\mu}_1 \cdot \vec{\mu}_2$, where r denotes the distance between the particles and $f(r)$ the r-dependent coupling constant between the spins. We expect $f(r) > 0$, which favors pairing of opposite spins. In order to separate such a pair to infinite distance, we need to overcome the attraction due to the opposite charges and the attraction due to the spin-spin interaction. Thus the separation energy for the particles with opposite spins is higher than for the particles without spins. Thus the critical temperature where infinitely separated vortex pairs break up increases with the increasing value of the spin. Since $|\vec{\mu}|\propto\alpha$ and thus $V\propto\alpha^2$ the critical temperature $T_c(\alpha)$ should grow as

$$T_c(\alpha)=T_c(0)+g\left(\frac{\alpha}{\pi}\right)^2, \tag{22}$$

where g is a constant and $T_c(0)$ is the critical temperature of the three-dimensional x-y model. This is what we indeed find in our numerical study of the model (21) as described in Sec. IV.

III. THE PHYSICAL OBSERVABLES AND MONTE-CARLO METHOD

The physical quantities we would like to compute for the model (21) are the specific heat c and the helicity modulus Υ_μ. The specific heat is obtained by

$$c=\beta^2\frac{\langle\mathcal{H}^2\rangle-\langle\mathcal{H}\rangle^2}{V}, \tag{23}$$

where V denotes the volume of the lattice. The helicity modulus is defined as follows:[21,22]

$$\frac{\Upsilon_\mu}{J}=\frac{1}{V}\left\langle\sum_{\langle i,j\rangle}\cos(\theta_i-\theta_j)(\vec{e}_\mu\cdot\vec{\epsilon}_{ij})^2\right\rangle -\frac{\beta}{V}\left\langle\left(\sum_{\langle i,j\rangle}\sin(\theta_i-\theta_j)\vec{e}_\mu\cdot\vec{\epsilon}_{ij}\right)^2\right\rangle, \tag{24}$$

$\vec{e}_\mu$ is the unit vector in the corresponding bond direction and $\vec{\epsilon}_{ij}$ is the vector connecting the lattice sites i and j. Since our system is isotropic we will omit the vector notation for the helicity modulus in the following.

The expectation values in (21) and (22) are computed with respect to the partition function Z given by the expression (21), i.e., the expectation value of a physical observable O is obtained according to

$$\langle O\rangle=Z^{-1}\int\prod_j d\theta_j O[\theta]\exp(-\beta\mathcal{H}+i\alpha N_{\rm CS}). \tag{25}$$

In order to compute expectation values (25) using the Monte Carlo method we proceed as in Ref. 23. The partition function given by Eq. (21) can be rewritten as follows:

$$Z=\sum_{N_{\rm CS}}\rho(N_{\rm CS})\exp(i\alpha N_{\rm CS}), \tag{26}$$

with $\rho(N_{\rm CS})$ being the density of the configurations whose value for the Chern-Simons number is $N_{\rm CS}$. This density is given by

$$\rho(N_{\rm CS})=\int\prod_i d\theta_i\bigg|_{N_{\rm CS}}\exp(-\beta\mathcal{H}). \tag{27}$$

The integration on the right-hand side of Eq. (27) includes only those configurations whose value for the Chern-Simons number is $N_{\rm CS}$. The expectation value (25) can then be expressed as

$$\langle O\rangle=Z^{-1}\sum_{N_{\rm CS}}\rho(N_{\rm CS})\langle O\rangle_{N_{\rm CS}}\exp(i\alpha N_{\rm CS}), \tag{28}$$

where

$$\langle O\rangle_{N_{\rm CS}}=\frac{1}{\rho(N_{\rm CS})}\int\prod_i d\theta_i\bigg|_{N_{\rm CS}}\exp(-\beta\mathcal{H}). \tag{29}$$

In a real computer simulation we are able to compute $\rho(N_{\rm CS})$, Z, and $\langle O\rangle_{N_{\rm CS}}$ to a sufficient accuracy. Therefore using (28) we can compute the average value $\langle O\rangle$. For our simulation we assume $\rho(N_{\rm CS})$ and $\langle O\rangle_{N_{\rm CS}}$ to be even functions with respect to $N_{\rm CS}$, i.e., all expectation values are real.

The simulation is done as follows. We generate a spin configuration according to the probability distribution

$$P[\theta]\propto\exp(-\beta\mathcal{H}), \tag{30}$$

by using Wolff's 1-cluster algorithm.[24] We measure the Chern-Simons number $N_{\rm CS}$ of this configuration, increment $\rho(N_{\rm CS})$ by 1 and add the value of the observable $O(N_{\rm CS})$ to a table containing all the values $O(N_{\rm CS})$ labeled by the Chern-Simons numbers. Next a new spin configuration is created. The histogram $\rho(N_{\rm CS})$ is proportional to the distribution (27), the arithmetic average of the values $O(N_{\rm CS})$ for a given value of $N_{\rm CS}$ yields an estimate for $\langle O\rangle_{N_{\rm CS}}$. This allows us to calculate the average value $\langle O\rangle$ given by (28). The accuracy of this method can be improved by using reweighting techniques.[25]

We computed the helicity modulus and the specific heat on lattices of sizes $L\times L\times L$ with $L=4$, 5, 6, 8, 10. Periodic boundary conditions were applied. We carried out of the order of 50 000 thermalization steps and of the order of 2 000 000 measurements. The computations were performed on a heterogeneous environment of workstations that include Sun, IBM RS/6000, and DEC alpha AXP workstations. Due to the reweighting procedure and the broadening of the densitiy distribution $\rho(N_{\rm CS})$ with larger lattices, runs on lattices with $L\gg 10$ become extremely time consuming. Therefore we restricted ourselves to the above lattice sizes.

IV. RESULTS

Let us first calculate the normalized probability distribution of the Chern-Simons number $P(N_{\rm CS})$ defined as

$$P(N_{\rm CS})=\frac{\rho(N_{\rm CS})}{\Sigma_{N_{\rm CS}}\rho(N_{\rm CS})}, \tag{31}$$

where $N_{\rm CS}$ and $\rho(N_{\rm CS})$ are defined by Eqs. (20) and (27). Figure 1 shows the distribution function $P(N_{\rm CS})$ for different lattice sizes and different temperatures. As can be seen $P(N_{\rm CS})$ is strongly affected by the size of the system and the temperature. Namely, by increasing the lattice size or the temperature we observe a broadening of the probability distribution $P(N_{\rm CS})$.

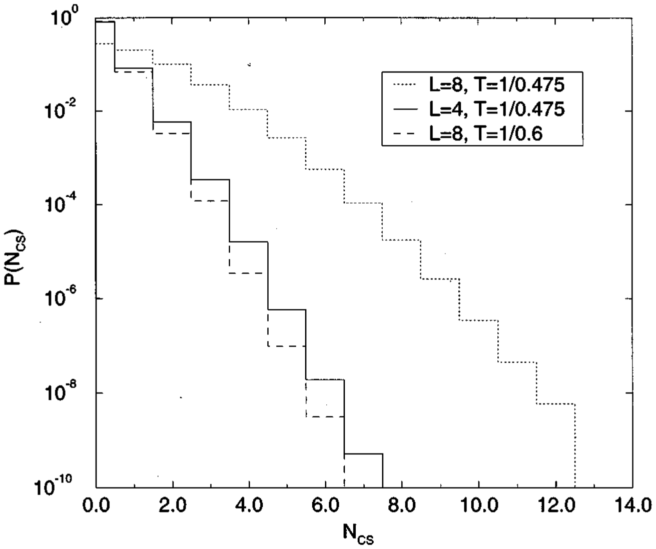

FIG. 1. The probability distribution $P(N_{\rm CS})$ for different lattice sizes and temperatures.

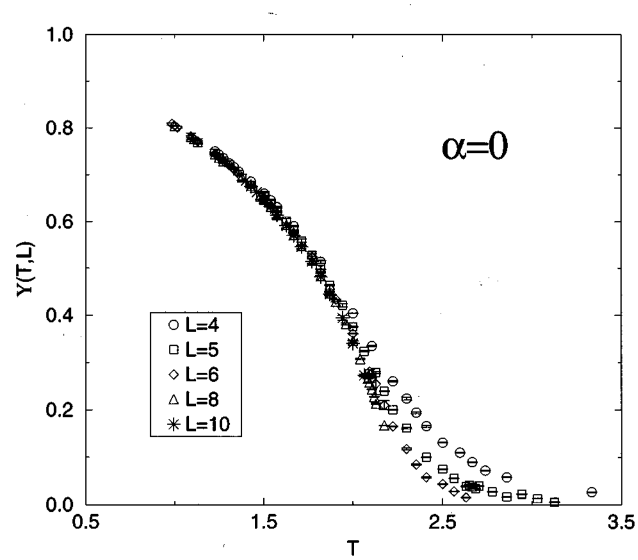

FIG. 2. The helicity modulus $\Upsilon(T,L)$ as a function of T, $\alpha=0$.

In order to create the Chern-Simons number $N_{\rm CS}$ a certain amount of energy $E_{N_{\rm CS}}$ is required. If we assume this energy to be independent of $N_{\rm CS}$ we can write for $P(N_{\rm CS})$:

$$P(N_{\rm CS})\propto\exp(-\beta E_{N_{\rm CS}}N_{\rm CS}). \tag{32}$$

Thus the slope $N_{\rm CS}^{-1}\ln P(N_{\rm CS})=-\beta E_{N_{\rm CS}}$. The constant slopes can be seen in Fig. 1 for large values of $N_{\rm CS}$. The energy $E_{N_{\rm CS}}$ itself depends on the lattice size L and the temperature.

A. The helicity modulus

Here we investigate the finite-size scaling behavior of the helicity modulus for different values of the parameter α. We use the finite-size scaling properties of the helicity modulus to extract an estimate for the α-dependent critical temperature $T_c(\alpha)$ and to study the α dependence of the critical exponent ν of the helicity modulus.

At this point we would like to mention one difficulty we encountered in the course of our computations. Because of the imaginary part in the partition function (21) the error bars of the computed quantities grow with increasing values of α. As long as the distribution $P(N_{\rm CS})$ is sharply peaked around $N_{\rm CS}=0$ the error bars are reasonably small. This happens for small temperatures and small lattice sizes (cf. Fig. 1 for the different shapes of $P(N_{\rm CS})$ and Figs. 2 and 3 for the size of the error bars). For larger temperatures and larger lattices $P(N_{\rm CS})$ grows broader and broader, which requires longer and longer simulation times in order to keep the error bars reasonably small. Therefore we restricted our computer simulations to lattices with at most 1000 lattice sites.

In Figs. 2 and 3 we plot our data for the helicity modulus with respect to temperature for $\alpha=0$ and $\alpha=\pi$, respectively. We see that the values of the helicity modulus corresponding to $\alpha=\pi$ are larger than the ones pertaining to $\alpha=0$ for temperatures $T\geqslant 1.7$. In general we find $\Upsilon(T,L,\alpha_1)\geqslant\Upsilon(T,L,\alpha_2)$ if $\alpha_1>\alpha_2$. For low temperatures the values of $\Upsilon(T,L)$ for both $\alpha=0$ and $\alpha=\pi$ are about the same. Thus the spin stiffness becomes larger when the vortices have a nonzero spin. Therefore we expect the critical temperature to grow with increasing α.

In order to find a rough estimate for the critical temperatures at various α we plot the dimensionless quantity $L\Upsilon(T,L,\alpha)/J$ versus T. Following the finite-size scaling theory[26] we expect to find that

$$\frac{L\Upsilon(T,L,\alpha)}{J}=f\left(\frac{L}{\xi(T)},\alpha\right) \tag{33}$$

close to the critical temperature $T_c(\alpha)$. $\xi(T)$ denotes the correlation length of the infinitely extended system and f is a universal function. At $T_c(\alpha)$ the correlation length is infinite, thus for fixed L the right-hand side of Eq. (33) is a constant.

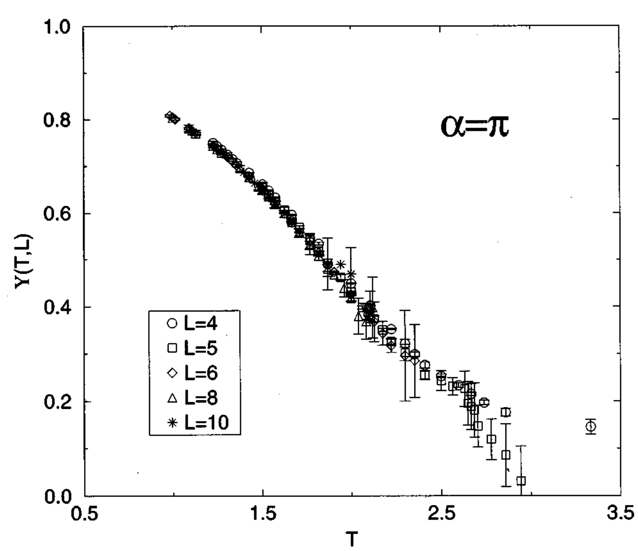

FIG. 3. The helicity modulus $\Upsilon(T,L)$ as a function of T, $\alpha=\pi$.

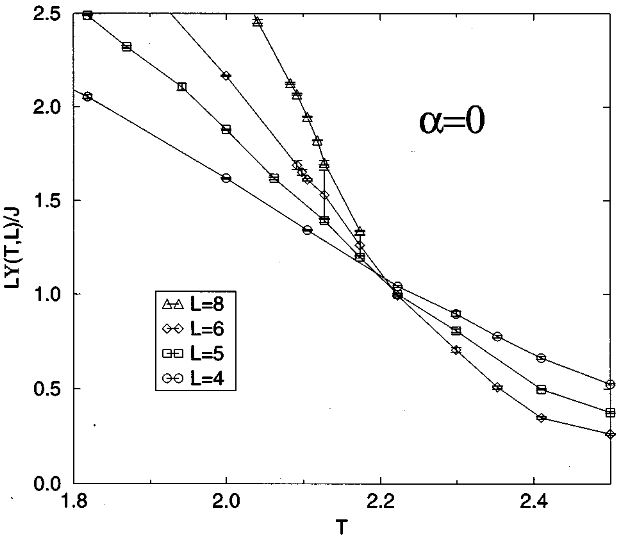

FIG. 4. $L\Upsilon(T,L)/J$ as a function of T, $\alpha=0$.

Therefore all the curves $L\Upsilon(T,L,\alpha)/J$ intersect in one point whose abscissa is $T_c(\alpha)$. In Figs. 4 and 5 we show $L\Upsilon(T,L,\alpha)/J$ vs T for $\alpha=0$ and $\alpha=\pi/2$, respectively. From the plots we read off $T_c(0)=2.206(12)$ and $T_c(\pi/2)=2.329(11)$. The error bars where estimated with respect to the scattering of the intersection points. Using the described method we obtained estimates of the critical temperatures for $\alpha=0,\pi/4,\pi/2,3\pi/4$ (cf. Table I). For $\alpha=\pi$ this method of determining T_c is not accurate enough due to the large error bars of $L\Upsilon(T,L)/J$. We will determine $T_c(\pi)$ approximately using a different technique as discussed further below.

In order to check if the critical exponent ν becomes α dependent let us explore formula (33) a little more. From the finite-size scaling theory we know that[26]

$$\xi(t\rightarrow 0)=A(\alpha)|t|^{-\nu(\alpha)}, \tag{34}$$

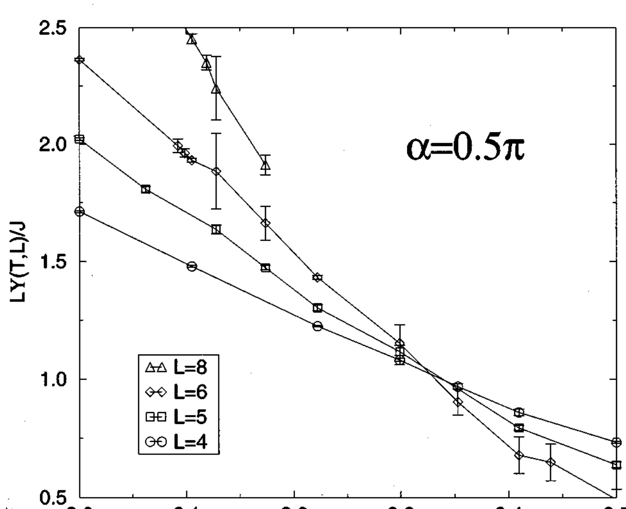

FIG. 5. $L\Upsilon(T,L)/J$ as a function of T, $\alpha=\pi/2$.

TABLE I. The critical temperature for different values of α.

α	$T_c(\alpha)$
0	2.206(12)
$\pi/4$	2.227(13)
$\pi/2$	2.329(11)
$3\pi/4$	2.52(8)
π	2.65(10)

where the reduced temperature $t=1-T/T_c(\alpha)$. We introduced a possible α dependence of the constant A and the critical exponent ν in formula (34). We have $\nu(0)=0.6705$.[27–29] With the help of (34) we may rewrite expression (33) as follows:

$$\frac{L\Upsilon(T,L,\alpha)}{J}=\tilde{f}(tL^{1/\nu(\alpha)},\alpha). \tag{35}$$

Equation (35) means that we obtain one universal curve for all values of L if we plot $L\Upsilon(T,L,\alpha)/J$ versus $tL^{1/\nu(\alpha)}$. This is indeed the case for $\alpha=0$, which is demonstrated in Fig. 6. The data collapse on one universal curve for $tL^{1/\nu(0)}>-1.6$, for the plot we used $T_c(0)=2.206$ (cf. Table I) and $\nu(0)=0.6705$. We are now in a position to check the α dependence of the critical exponent ν. If the inequality $\nu(\alpha)\neq 0.6705$ is true we will not find a universal curve by plotting $L\Upsilon(T,L,\alpha)/J$ vs $tL^{1/0.6705}$. In Fig. 7 we demonstrate the collapse of our data points for $\alpha=\pi/2$ with $T_c(\pi/2)=2.329$ (cf. Table I) and $\nu(\pi/2)=0.6705$. We found this collapse of our data also for $\alpha=\pi/4$ and $\alpha=3\pi/4$ when the value of $\nu=0.6705$ was taken. So far our results suggest that the critical exponent ν be independent of the parameter α.

In the model considered by Chen, Fischer, and Wu [11] the critical exponent ν depends on α and covers the range $[0.6705,1]$ for $\alpha\in[0,\pi]$. In particular for $\alpha=3\pi/4$ we

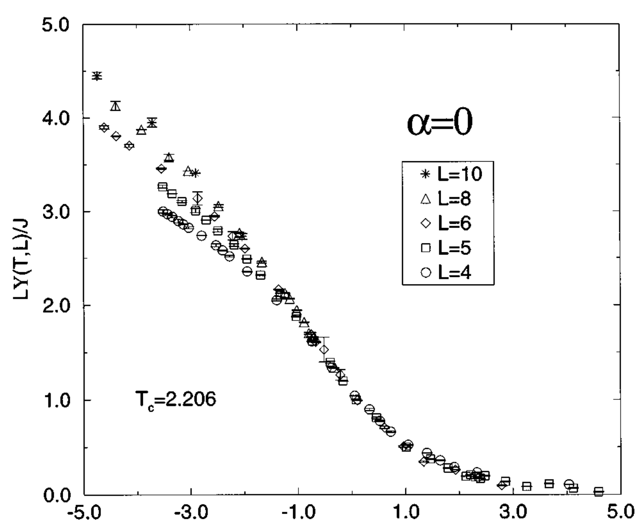

FIG. 6. $L\Upsilon(T,L)/J$ as a function of $tL^{1/\nu}$ for $\alpha=0$, $\nu=0.6705$, $T_c=2.206$.

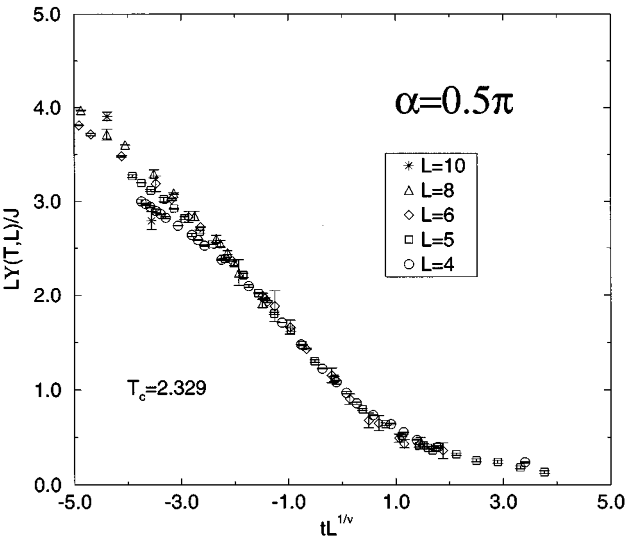

FIG. 7. $L\Upsilon(T,L)/J$ as a function of $tL^{1/\nu}$ for $\alpha=\pi/2$, $\nu=0.6705$, $T_c=2.329$.

should have $\nu=0.85$.[11] In order to compare scaling of the helicity modulus with $\nu=0.85$ and $\nu=0.6705$, respectively, we employ the scaling form:

$$\frac{\Upsilon(T,L,\alpha)}{J}|t|^{-\nu}=g(tL^{1/\nu},\alpha), \tag{36}$$

where $g(x,\alpha)=x^{-\nu}\tilde{f}(x,\alpha)$. This scaling form is more sensitive to the value of ν than the expression (35). From Eq. (36) we conclude that in the limit $\xi(t)\leq L$, i.e., in the bulk critical limit, the left-hand side of Eq. (36) becomes a constant. In this case the data collapse onto a straight line. In Figs. 8 and 9 we show the corresponding scaling plots for $\alpha=3\pi/4$ where we used $\nu=0.6705$ and 0.85, respectively. The values for the helicity modulus collapse onto a straight

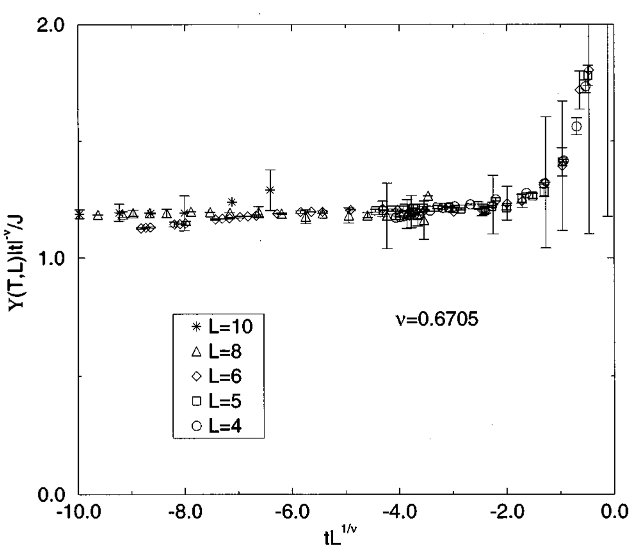

FIG. 8. $|t|^{-\nu}\Upsilon(T,L)/J$ as a function of $tL^{1/\nu}$ for $\alpha=3\pi/4$, $\nu=0.6705$, $T_c=2.52$.

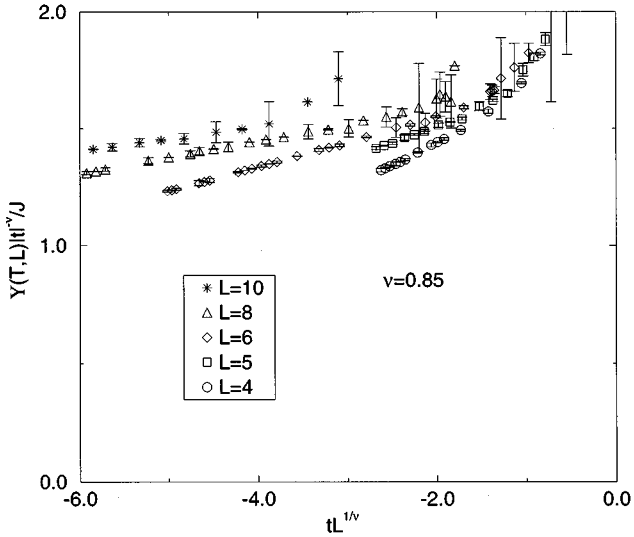

FIG. 9. $|t|^{-\nu}\Upsilon(T,L)/J$ as a function of $tL^{1/\nu}$ for $\alpha=3\pi/4$, $\nu=0.85$, $T_c=2.52$.

line when $\nu=0.6705$ is used, whereas there is no scaling for $\nu=0.85$. Furthermore, a slight variation of the value of ν around $\nu=0.6705$ revealed that the data collapse was possible within error bars for $\nu=0.67\pm0.07$. If in our model the critical exponent ν depends on α at all then this dependence is rather weak, the possible values for $\nu(\alpha)$ do not cover the range $[0.6705,1]$ for $\alpha\in[0,\pi]$. We are not surprised that we obtain a different result than do Chen, Fisher, and Wu[11] as our model and the one considered in Ref. 11 are not equivalent. The difference lies in the way we incorporate the Chern-Simons term into the theory as was outlined in the Introduction. In our theory (21) the statistics changing gauge field is generated by the topological current causing the nonlocality of the Chern-Simons term (compare also Ref. 13). The Chern-Simons gauge field in the action used by Chen, Fisher, and Wu[11] is a truly fluctuating field minimally coupled to the complex order parameter. We only mention here that the two actions are equivalent if the order parameter has at least two complex components.[15] The argument employed by the authors of Ref. 15 does not work in our case of a one-component complex order parameter.

Since our results indicate that ν is not affected when the x-y model is coupled to the Chern-Simons term defined in Sec. II according to (21) we are able to estimate the critical temperature $T_c(\pi)$, namely, we plot $L\Upsilon(T,L)/J$ vs $tL^{1/0.6705}$ but vary the value for $T_c(\pi)$ until our data points collapse onto one universal curve. This can be achieved reasonably well for $T_c(\pi)=2.65(10)$. The scaling function thus obtained is displayed in Fig. 10. We notice that the range of $tL^{1/0.6705}$ where the data collapse on one universal curve seems larger for larger α.

In Fig. 11 we notice that the critical temperature $T_c(\alpha)$ increases monotonically with increasing α. In order to check whether our values for $T_c(\alpha)$ are consistent with the expression (22) we fitted the functional form (22) to the critical temperature values given in Table I with the fixed parameter $T_c(0)=2.206$. We obtained $g=0.484\pm0.038$ and the fit is given by the parabola in Fig. 11. Since the parabola fits the computed values for $T_c(\alpha)$ rather well, we can think of the

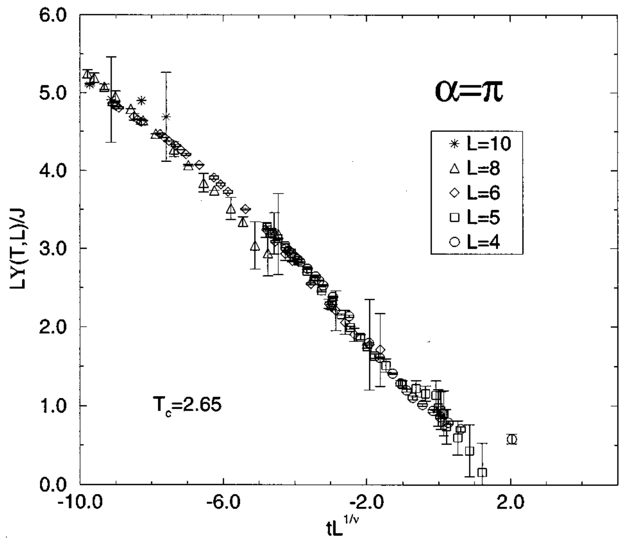

FIG. 10. $LY(T,L)/J$ as a function of $tL^{1/\nu}$ for $\alpha=\pi$, $\nu=0.6705$, $T_c=2.65$.

x-y model with the Chern-Simons term (20) as describing vortices with spin $\pm\alpha/(2\pi)$ moving in Euclidean time, which interact through the potential due to the vortex charges and the spin-spin interaction potential $\propto\alpha^2$.

B. The specific heat

In this section we briefly comment on the behavior of the specific heat with respect to the spin coefficient α.

Since the critical exponent of the correlation length ν does not depend on α (cf. the previous section) the critical exponent of the specific heat $\tilde{\alpha}$ cannot depend on α either due to the hyperscaling assumption $\tilde{\alpha}=2-3\nu$. We denote the critical exponent of the specific heat by $\tilde{\alpha}$ instead of α (which is commonly used in the literature to denote the specific-heat critical exponent) in order to avoid confusion with the parameter α of this paper [cf. Eq. (21)].

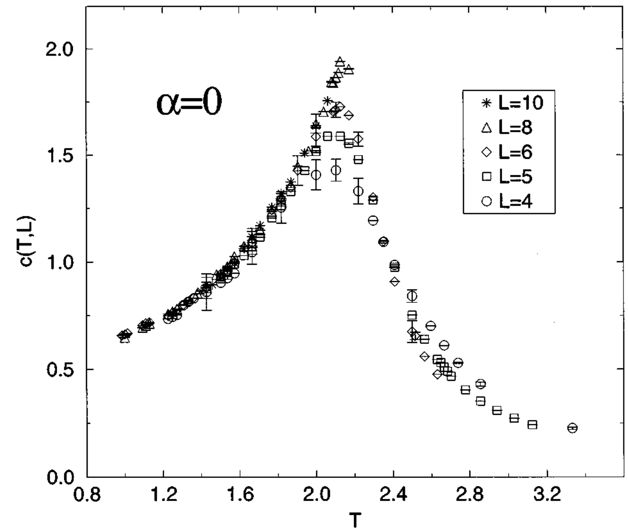

FIG. 12. The specific heat for various lattice sizes as a function of temperature at $\alpha=0$.

Figures 12 and 13 show the specific heat for various lattice sizes at $\alpha=0$ and $\alpha=\pi/2$, respectively. At low temperatures the specific-heat data agree for all lattice sizes and all values of α. This should be the case since at these temperatures the correlation length is smaller than the lattice size L, thus the specific heat does not feel the finite size of the system. Furthermore the probability of finding the Chern-Simons number $N_{\rm CS}\neq 0$ is very small, i.e., the parameter α does not have any influence on the physical quantities at all. However, for temperatures closer to the critical temperature $T_c(0)$ we find that the values of the specific heat become smaller with increasing α. This effect is independent of the system size L (cf. Figs. 12 and 13). Our findings seem to indicate that the finite value of the specific heat $c[T_c(\alpha),L=\infty]$ is α dependent and decreases with increasing α. However, in order to determine these values to a

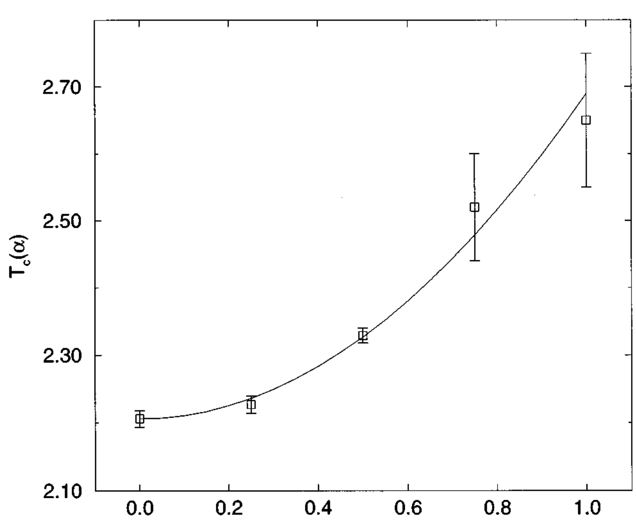

FIG. 11. The critical temperatures $T_c(\alpha)$ as a function of α. The solid line represents Eq. (22) with $T_c(0)=2.206$ and $g=0.484$.

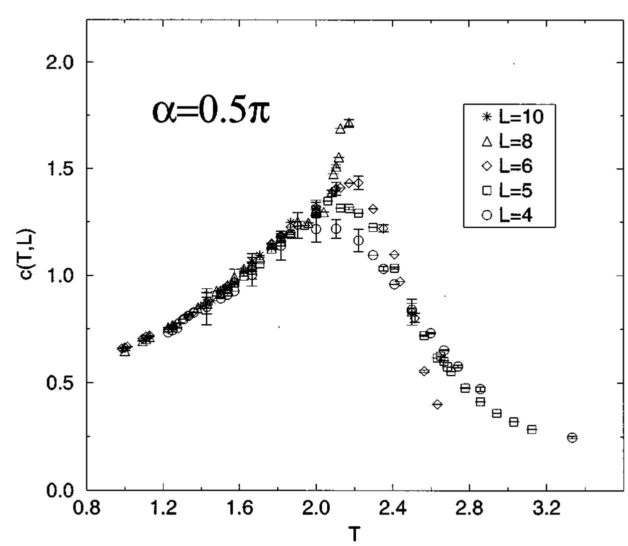

FIG. 13. The specific heat for various lattice sizes as a function of temperature at $\alpha=\pi/2$.

sufficient accuracy, computations on much larger lattices have to be performed, which require very long computation times due to the broadening of the probability distribution $P(N_{\rm CS})$.

V. SUMMARY

We have coupled the three-dimensional x-y model to the Chern-Simons term in the spirit of Wilczek and Zee,[13] thus endowing the vortices with a spin $\alpha/(2\pi)$. We have investigated the influence of the Chern-Simons term on the critical behavior. The geometrical interpretation of this term enabled us to compute the Chern-Simons number for an arbitrary spin configuration on the lattice. We computed the helicity modulus and the specific heat on $L\times L\times L$ lattices up to $L=10$ by means of a Monte Carlo simulation. Periodic boundary conditions were applied in all directions. We used the finite-size scaling properties of the helicity modulus to estimate the critical temperatures and to check the influence of the Chern-Simons term on the critical exponent ν. Our findings suggest that the x-y model coupled to the Chern-Simons term following Wilcek and Zee[13] belongs to the x-y universality class, however, the critical temperature and the finite bulk value of the specific heat at the critical temperature depend on α.

ACKNOWLEDGMENTS

We would like to thank S. Kivelson for his valuable comments on the first version of our paper. N.S. enjoyed discussions with L. Pryadko and would like to thank D. W. Sumners and E. Klassen for discussing the form language and the nature of Chern-Simons terms. This work was supported by the National Aeronautics and Space Administration under Grant No. NAGW-3326.

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